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# Correlators of Matrix Models
# on Homogeneous Spaces

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## Abstract

We investigate the correlators of $TrA_\mu A_\nu$ in matrix models on homogeneous spaces: $S^2$ and $S^2 \times S^2$. Their expectation value is a good order parameter to measure the geometry of the space on which non-commutative gauge theory is realized. They also serve as the Wilson lines which carry the minimum momentum. We develop an efficient procedure to calculate them through 1PI diagrams. We determine the large $N$ scaling behavior of the correlators. The order parameter shows that fuzzy $S^2 \times S^2$ acquires a 4 dimensional fractal structure in contrast to fuzzy $S^2$. We also find that the two point functions exhibit logarithmic scaling violations.

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# 1　Introduction

One of the most important problems in string theory is to understand the dynamics of D-branes. The understanding of such a problem is likely to lead to the ultimate formulation of string theory. For example, old matrix models for non critical strings are reinterpreted as matrix models of D-branes[1][2].

Such a development reinforces our interest in matrix models for critical strings[3][4]. In matrix models, non-commutative solutions are identified as D-branes. On such a solution, non-commutative (NC) gauge theory is realized[5][6][7]. In string theory, NC gauge theory arises on D-branes with $B_{\mu\nu}$ field[8]. More recently the non-commutativity parameter is identified with $g_s\alpha'$ where $g_s$ is the string coupling constant and $\alpha'$ is the inverse string tension[9][10]. We therefore suspect that non-commutativity plays a fundamental role in string theory[11].

In a non-perturbative investigation, it is desirable to work with matrices of finite $N$. In such a setting, branes are inevitably compact. The simplest such objects in matrix models are fuzzy sphere and its higher dimensional analogues[12][13][14]. In order to obtain such a classical solution in matrix models with finite $N$, we need to introduce Myers term and its generalizations[15][16][17]. It serves as a nonperturbative formulation of NC gauge theories on homogeneous spaces $G/H$.

We have investigated quantum corrections of NC gauge theory on $G/H$[18]$\sim$[21]. We have identified the 't Hooft couplings and large $N$ scaling behavior of these gauge theories by power counting arguments. Recently a non-perturbative study has been performed in a bosonic matrix model[22]. The large $N$ scaling behavior of the observables are found to be in good agreement with perturbative predictions.

From string theory point of view, the Myers term corresponds to the constant $H_{\mu\nu\rho}$ field. With finite $N$, most of the solutions are metastable since the single $S^2$ solution always minimizes the action at the classical level. Nevertheless we expect that they are stabilized in the large $N$ limit due to the suppression of the tunneling effect. We have indeed found that the $H_{\mu\nu\rho}$ field vanishes in our large $N$ scaling analysis. We therefore argue that various $G/H$ solutions approach non-commutative $R^d$ with various dimensionality $d$ in the large $N$ limit realizing the formal classical solutions of IIB matrix model. Our investigations concern quantum fluctuations around a particular extremum of the effective action and the exploring the entire landscape is beyond the scope of this paper.

Another possibility is to contemplate that such a solution extremizes the quantum effective action of IIB matrix model. In fact we have found that a 4 dimensional solution of $S^2 \times S^2$ type extremizes the effective action at the two loop level[19][21]. Since its effective action is $O(N)$ which is the same order with the space-time volume, we may interpret this solution as a D3 brane with a finite tension. It is because the true minimum of the effective action is argued to be $O(1)$ in IIB matrix model[23][24].

In the matrix model construction of NC gauge theory, space-time and gauge field are unified into the identical matrix degrees of freedom $A_\mu$ which are identified as the coordinates semiclassically. In this sense the gauge invariant operator $Tr A_\mu A_\nu$ is a good order parameter to measure the shape and extension of the Euclidean space-time on which NC gauge theory is realized.

In this paper, we investigate the vacuum expectation value (one point function) of these operators in detail. We develop an efficient procedure to calculate them through 1PI diagrams. We perform explicit calculations on $S^2$ and $S^2 \times S^2$ in various matrix models up to the two loop level. We identify the large $N$ scaling behavior of the correlators to all orders by power counting arguments. We find that the correlators on $S^2$ and $S^2 \times S^2$ exhibit different scaling behavior due to different dimensionality. In particular $< Tr A_\mu A_\nu >$ on $S^2 \times S^2$ indicates a 4d fractal like structure of it in contract to $S^2$ which remains to be semiclassical.

The organization of this paper is as follows. In section 2, we investigate the effective action of quantum fields at the one loop level in preparation to calculate the correlators at the two loop level. We discuss how gauge invariance is respected in NC gauge theory. In section 3, we calculate the vacuum expectation value of this operator on $S^2$ up to the two loop level. We first evaluate it by the connected diagrams. We then evaluate it by 1PI diagrams after expanding $A_\mu$ around the quantum solution which extremizes the effective action. We find that the both methods produce the identical answer as it can be argued on general grounds. The order parameter is found to consist of the Kronecker's $\delta$ function of 3d space in which $S^2$ extends. In section 4, we calculate the corresponding order parameter on $S^2 \times S^2$ by 1PI diagrams. We find that it consists of the two independent tensors. We interpret the new tensor as the contribution from 4d fractals. In section 5, we explain that these operators also serve as the Wilson lines on homogeneous spaces[25][26]. We compute the connected two point functions of these Wilson lines. We conclude in section 6 with discussions.

## 2 Effective action

In this section, we investigate the one loop self-energy in NC gauge theory on $S^2$. This investigation is useful to compute the gauge invariant correlators up to the two loop level. Since NC gauge theory is realized by matrix models with finite $N$, the gauge invariance is exactly maintained. Our first concern in this section is how the gauge invariance is reflected by the one loop self-energy of gauge fields.

Let us consider IIB matrix model:

$$S_{IIB} = -\frac{1}{g^2} Tr(\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]). \tag{2.1}$$

We expand $A_\mu$ around a background $(p_\mu)$ which represents a fuzzy $S^2$:

$$A_\mu = f(p_\mu + a_\mu). \tag{2.2}$$

where $a_\mu$ denotes the gauge field on it. $p_\mu$ can be identified with the angular momentum operators in the spin $l$ representation where $N = n(2l+1)$ in the case of $n$ identical spheres. $f$ is the scale factor of this background.

At the one loop level, the effective action consists of the gauge sector (gauge field and ghost) and the fermion sector contributions. The both contributions must be BRS invariant and the fermion sector contribution must be gauge invariant in particular. The fermionic contribution to the one loop effective action is

$$-\frac{1}{2}Trlog(1 + \Gamma \cdot \delta A \frac{1}{\Gamma \cdot P})$$
$$= -\frac{1}{2}Tr[\Gamma \cdot \delta A \frac{1}{\Gamma \cdot P}] + \frac{1}{4}Tr[\Gamma \cdot \delta A \frac{1}{\Gamma \cdot P}\Gamma \cdot \delta A \frac{1}{\Gamma \cdot P}] + \cdots, \tag{2.3}$$

where $\delta A_\mu X = [a_\mu, X]$. In this expression, the first and second terms represent the tadpole and self-energy of gauge field respectively.

The gauge field self-energy (two point function) in the above expression consists of the planar and non-planar contributions:

$$-\frac{1}{2} < a^\mu|tr\Gamma_\mu \frac{1}{\Gamma \cdot Q}\Gamma_\nu \frac{1}{\Gamma \cdot R}|a^\nu >_p$$
$$+\frac{1}{2} < a^\mu|tr\Gamma_\mu \frac{1}{\Gamma \cdot Q}\Gamma_\nu \frac{1}{\Gamma \cdot R}|a^\nu >_{np}, \tag{2.4}$$

where the symbol $tr$ evaluates the trace over spinor indices only and

$$P_\mu Y_{j_1 m_1} \equiv [p_\mu, Y_{j_1 m_1}],$$
$$Q_\mu Y_{j_2 m_2} \equiv [p_\mu, Y_{j_2 m_2}],$$
$$R_\mu Y_{j_3 m_3} \equiv [p_\mu, Y_{j_3 m_3}]. \tag{2.5}$$

We have also introduced the following average:

$$
\begin{aligned}
< a^\mu |X| a^\nu >_p &= \sum_{j_2, j_3, m_2, m_3} \Psi^*_{a^\mu 23} X \Psi_{a^\nu 23}, \\
< a^\mu |X| a^\nu >_{np} &= \sum_{j_2, j_3, m_2, m_3} \Psi^*_{a^\mu 32} X \Psi_{a^\nu 23}, \\
\Psi_{a^\mu 23} &\equiv \sum_{j_1 m_1} a^\mu_{j_1 m_1} Tr Y_{j_3 m_3} Y_{j_2 m_2} Y_{j_1 m_1}.
\end{aligned}
\tag{2.6}
$$

For simplicity, we restrict our consideration to $U(1)$ gauge theory here as it is straightforward to generalize it to $U(n)$ gauge theory.

Under the gauge transformation:

$$
a_\mu \to P_\mu \Lambda + [a_\mu, \Lambda],
\tag{2.7}
$$

the self-energy (2.4) changes as

$$
\begin{aligned}
&\frac{1}{2} < \Lambda | tr \Gamma \cdot P \frac{1}{\Gamma \cdot Q} \Gamma_\nu \frac{1}{\Gamma \cdot R} | a^\nu >_p \\
&- \frac{1}{2} < a^\mu | tr \Gamma_\mu \frac{1}{\Gamma \cdot Q} \Gamma \cdot P \frac{1}{\Gamma \cdot R} | \Lambda >_p \\
= \ &- \frac{1}{2} < \Lambda | tr \frac{1}{\Gamma \cdot Q} \Gamma_\nu | a^\nu >_p - \frac{1}{2} < \Lambda | tr \frac{1}{\Gamma \cdot R} \Gamma_\nu | a^\nu >_p \\
&+ \frac{1}{2} < a^\nu | tr \frac{1}{\Gamma \cdot Q} \Gamma_\nu | \Lambda >_p + \frac{1}{2} < a^\nu | tr \frac{1}{\Gamma \cdot R} \Gamma_\nu | \Lambda >_p \\
= \ &-8 \sum_{j_2, m_2} Tr Y^\dagger_{j_2 m_2} \left( \frac{Q_\nu}{Q^2} Y_{j_2 m_2} \right) [a^\nu, \Lambda] \\
&+8 \sum_{j_2, m_2} Tr \left( \frac{Q_\nu}{Q^2} Y_{j_2 m_2} \right) Y^\dagger_{j_2 m_2} [a^\nu, \Lambda],
\end{aligned}
\tag{2.8}
$$

where we have retained only the linear terms in $a$. The non-planar part is gauge invariant by itself at the linearized level.

Since the fermionic contribution to the effective action must be gauge invariant, it is somewhat surprising to find that the gauge field self-energy is not gauge invariant by itself. The resolution of this puzzle is that the gauge invariance is restored by the presence of the tadpole. The tadpole (one point function) is

$$
\begin{aligned}
&-\frac{1}{2} Tr \Gamma \cdot \delta A \frac{1}{\Gamma \cdot Q} \\
= \ &8 \sum_{j_2, m_2} Tr Y^\dagger_{j_2 m_2} \left( \frac{Q_\mu}{Q^2} Y_{j_2 m_2} \right) a^\mu \\
&-8 \sum_{j_2, m_2} Tr \left( \frac{Q_\mu}{Q^2} Y_{j_2 m_2} \right) Y^\dagger_{j_2 m_2} a^\mu.
\end{aligned}
\tag{2.9}
$$

4

The tadpole changes under gauge transformation as

$$8 \sum_{j_2, m_2} Tr Y_{j_2 m_2}^\dagger \Big( \frac{Q_\mu}{Q^2} Y_{j_2 m_2} \Big) [a^\mu, \Lambda]$$

$$-8 \sum_{j_2, m_2} Tr \Big( \frac{Q_\mu}{Q^2} Y_{j_2 m_2} \Big) Y_{j_2 m_2}^\dagger [a^\mu, \Lambda]. \tag{2.10}$$

Since (2.8) and (2.10) cancel each other, the fermionic contribution to the one loop effective action is explicitly shown to be gauge invariant at the linearized level. The lesson we draw here is that the transversality of the gauge field self-energy could be spoiled in NC gauge theory under the presence of the tadpole.

We next investigate the gauge sector contribution. In this section, we deform the IIB matrix model action by adding a Myers term in such a way that $S^2$ is a classical solution.

$$S_{IIB} + \frac{i}{3} f \epsilon_{\mu\nu\rho} Tr[A_\mu . A_\nu] A_\rho. \tag{2.11}$$

Since IIB matrix model is a large $N$ reduced model of 10d super Yang-Mills theory, we also evaluate the gauge sector contributions in the large $N$ reduced model of $D$ dimensional super Yang-Mills theory with the identical deformation.

Firstly we evaluate the tadpole as:

$$(D-2) Tr \delta A \cdot Q \frac{1}{Q^2}, \tag{2.12}$$

where $D-1$ and $-1$ are the gauge field and ghost contributions respectively. It precisely cancels the fermionic contribution (2.9) when the physical degrees of freedom coincide (in supersymmetric reduced models).

The gauge field sector contribution to the planar part of the gauge field self-energy is evaluated in Appendix A as

$$\frac{1}{2} < a^\mu | \frac{1}{Q^2 R^2} \Big( 4 P_\mu P_\nu - 4 P^2 \delta_{\mu\nu} - 8 \tilde{\delta}_{\mu\nu} + 8 i \epsilon_{\mu\nu\rho} P^\rho$$

$$+ (D-2)(Q_\mu R_\nu + Q_\nu R_\mu - Q_\nu Q_\mu - R_\nu R_\mu + (Q^2 + R^2) \delta_{\mu\nu}) \Big) | a^\nu >_p, \tag{2.13}$$

where $\tilde{\delta}_{\mu\nu}$ is Kronecker's $\delta$ in the 3 dimensional sub-space in which $S^2$ extends.

Let us consider the $D = 3$ case which is relevant to a recent Monte-Carlo investigation[22]. Firstly the one loop self-energy is of the same order with the tree action if we fix $f^4 N$ in the large $N$ limit. It is because (2.13) is $O(1/N)$ due to the $6j$ symbols in the interaction vertices. Secondly the self-energy is certainly positive for gauge fields which carry small

5

angular momentum. In the case of $n$ identical branes with $U(n)$ gauge group, there are zero-modes $a_0^\mu$. They represent the relative center of mass positions of the branes. For zero-modes, the gauge field self-energy can be estimated as

$$\frac{n^2}{2} tr < a^\mu | \frac{1}{Q^2 R^2} \left( \frac{2}{3} Q^2 \delta_{\mu\nu} - 8\delta_{\mu\nu} \right) | a^\nu >_p$$

$$\sim \frac{n^2}{2N} \left( \frac{4}{3} log(N/n) - 8 \right) tr a_0^\mu a_0^\mu, \tag{2.14}$$

where $tr$ is over the $n$ dimensional subspace of zero-modes. We find that there arises positive mass term for zero-modes in the large $N$ limit. Therefore the degeneracy is lifted at the one loop level. [4]

The fermionic contribution to the planar part of the gauge field self-energy (2.4) can be evaluated as

$$-\frac{16}{2} < a^\mu | \frac{1}{Q^2 R^2} \left( Q_\mu R_\nu + Q_\nu R_\mu - Q \cdot R \delta_{\mu\nu} - 2\tilde{\delta}_{\mu\nu} + \delta_{\mu\nu} + i\epsilon_{\mu\nu\rho} P^\rho \right) | a^\nu >_p . \tag{2.15}$$

After combining (2.13) and (2.15), the planar part of the one loop self-energy of gauge field in a deformed IIB matrix model is found as

$$\frac{1}{2} < a^\mu | \frac{1}{Q^2 R^2} \left( 4P^2 \delta_{\mu\nu} - 4P_\mu P_\nu + 8(\tilde{\delta}_{\mu\nu} - i\epsilon_{\mu\nu\rho} P^\rho) - 16(\delta_{\mu\nu} - \tilde{\delta}_{\mu\nu}) \right) | a^\nu >_p . \tag{2.16}$$

It is invariant under the linearized gauge transformation $\delta a_\mu = P_\mu \Lambda$ just like the self energy of gauge field in ordinary gauge theory. However it contains mass and Chern-Simon type terms. After inspecting (2.16), we note that the one-loop self energy is positive for the components of the gauge field in the subspace in which $S^2$ extends. However it is negative for the components in the extra-dimensions as long as their angular momentum is small. They are always interpreted as scalars from gauge theory on $S^2$ point of view. In the case of $n$ identical branes, (2.16) is consistent with the one loop effective action for zero-modes [18].

# 3   Correlators on $S^2$

In this section, we evaluate the expectation values of the following gauge invariant operator in NC gauge theory on $S^2$:

$$< \frac{1}{N} Tr A_\mu A_\nu > . \tag{3.1}$$

Firstly we evaluate it through connected diagrams expanding $A_\mu$ around a classical solution. We then evaluate it through 1PI diagrams after expanding $A_\mu$ around the quantum solution

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[4]With finite $N$, it could become negative leading to an instability[22].

which extremizes the effective action. We find that the both methods give the identical result as it can be argued to be the case. Based on the explicit calculations up to the two loop level, we identify the large $N$ scaling behavior of this correlator. We further argue that this scaling behavior is valid to all orders.

At the tree level, (3.1) is evaluated as

$$\frac{f^2}{N}Tr p_\mu p_\nu = \frac{f^2}{3}l(l+1)\tilde{\delta}_{\mu\nu} \sim \frac{f^2 N^2}{12n^2}\tilde{\delta}_{\mu\nu}. \tag{3.2}$$

It is consistent with the fact that the space is classically a fuzzy sphere of the radius $fl$. In the classical limit this operator measures the shape and extension of a space realized in a matrix model. It is hence a natural operator to measure the geometry of the space at the quantum level as well.

At the one loop level,

$$\frac{f^2}{N} < Tr a_\mu a_\nu >= \frac{n^2}{f^2 N}\sum_1^{2l}\frac{2j+1}{j(j+1)}\delta_{\mu\nu} \sim \frac{2n^2}{f^2 N}log(N/n)\delta_{\mu\nu}. \tag{3.3}$$

In a $D$ dimensional bosonic reduced model, we also obtain the following tadpole contribution due to the presence of the one point function (2.12):

$$\frac{f^2}{N} < Tr p_\mu a_\nu + a_\mu p_\nu >= -\frac{D-2}{3f^2}N\tilde{\delta}_{\mu\nu}. \tag{3.4}$$

It is of the same order with the tree contribution (3.2) if we fix $n^2/f^4 N$ while (3.3) is suppressed by $log(N)/N^2$ in the large $N$ limit. Thus the quantum effects tend to shrink the radius of the sphere. This effect is caused by the logarithmic attractive potential between eigenvalues in bosonic models. In fact $n^2/f^4 N$ cannot be too large since otherwise (3.4) overwhelms (3.2) leading to unphysical results.

It is an indication of an instability of the sphere when the coupling $n^2/f^4 N$ is strong enough. When $D = 3$, the stability of a fuzzy sphere solution can be estimated by the following one loop effective action with $< A_\mu >= \beta p_\mu$:

$$(\frac{1}{2}\beta^4 - \frac{2}{3}\beta^3 f)Nl(l+1) + N^2 log\beta + N^2 log(N/n). \tag{3.5}$$

The quantum vacuum is determine by minimizing it with respect to $\beta$:

$$\frac{1}{2n^2}(\beta^3 - \beta^2 f)N^3 + N^2\frac{1}{\beta} = 0. \tag{3.6}$$

In the weak coupling limit ($\beta \to \infty$), this equation reproduces the classical solution $\beta = f$. The deviation from the classical solution can be estimated from (3.6) to the leading order of $n^2/f^4N$ as

$$1 - \frac{f}{\beta} = -2\frac{n^2}{f^4N} + \cdots. \tag{3.7}$$

The order parameter is estimated as

$$\frac{\beta^2}{N}Trp_\mu p_\nu = \frac{f^2N^2}{12n^2}\Big(1 - 4\frac{n^2}{f^4N} + \cdots\Big)\tilde{\delta}_{\mu\nu}, \tag{3.8}$$

which is consistent with (3.4). We further note that the minimum of the effective action no longer exists when

$$\frac{n^2}{f^4N} > \frac{2^9}{3^3} \tag{3.9}$$

which indicates the instability of a fuzzy $S^2$ solution beyond this coupling constant. [5] In fact this argument has been found to be in good agreement with a Monte Carlo simulation[22].

Here we return to a supersymmetric model: the deformed IIB matrix model (2.11). At the two loop level, we obtain from the one loop effective action (2.16)

$$\frac{f^2}{N} < Tra_\mu a_\nu >$$
$$= -\frac{n^3}{f^6N} < \frac{1}{(P^2)^2Q^2R^2}\Big(4P^2\delta_{\mu\nu} - \frac{4}{3}P^2\tilde{\delta}_{\mu\nu} + 8\tilde{\delta}_{\mu\nu} - 16(\delta_{\mu\nu} - \tilde{\delta}_{\mu\nu})\Big) >_p + n.p., \tag{3.10}$$

where

$$
\begin{aligned}
< X >_p &= \sum_{j_1,j_2,j_3,m_1,m_2,m_3} \Psi^*_{123}X\Psi_{123}, \\
\Psi_{123} &= TrY_{j_3m_3}Y_{j_2m_2}Y_{j_1m_1}.
\end{aligned} \tag{3.11}
$$

The symbol $n.p.$ denotes the non-planar contributions. Each contribution in (3.10) is suppressed by $1/N^2$ compared to the tree contribution when we fix $n^2/f^4N$ in the large $N$ limit. Although there are no tadpoles at the one loop level in supersymmetric models, they do arise at the two loop level. The total tadpole contribution to this order parameter at the two loop level is evaluated in Appendix B as

$$\frac{f^2}{N} < Trp_\mu a_\nu + a_\mu p_\nu >= -\frac{n^3}{f^6N}\tilde{\delta}_{\mu\nu}\frac{172}{3} < \frac{1}{P^2Q^2R^2} >_p + n.p.. \tag{3.12}$$

---

[5]This critical coupling was first estimated by D. O'Connor.

The existence of the tadpole indicates the shift of the classical solution due to the quantum effects. Such a shift should also be calculable by minimizing the two loop effective action around $A_\mu = \beta p_\mu$. We can read off the relevant effective action from our previous results[18][19] as

$$
\begin{aligned}
&(\frac{1}{2}\beta^4 - \frac{2}{3}\beta^3 f)\frac{N^3}{4n^2} \\
&- \frac{n^3}{\beta^4}\Big(32 + 4(\frac{f^2}{\beta^2}) - 12(1 - \frac{f}{\beta}) - 32(1 - \frac{f}{\beta}) + 24(1 - \frac{f}{\beta})\Big) < \frac{1}{P^2 Q^2 R^2} >_p + n.p.,
\end{aligned}
\tag{3.13}
$$

where the second line represents the two loop contribution. The second term proportional to $(f^2/\beta^2)$ denotes the contribution from the Myers vertices. The factor $(1 - f/\beta)$ represents the modification of the gauge field propagator from the minimal one. In this expression, we have retained only the linear contributions with respect to this factor. The third and fourth terms come from gauge and fermionic contributions and the last term comes from the Myers and cubic gauge vertices. [6]

The deviation from the classical solution can be estimated by minimizing (3.13) with respect to $\beta$ to the leading order of $n^2/f^4 N$ as

$$
1 - \frac{f}{\beta} = -(\frac{n^5}{f^8 N^3}) < \frac{344}{P^2 Q^2 R^2} >_p + n.p..
\tag{3.14}
$$

We thus find that

$$
\frac{\beta^2}{N}Tr p_\mu p_\nu = (1 - (\frac{n^5}{f^8 N^3}) < \frac{688}{P^2 Q^2 R^2} >_p )\frac{f^2 N^2}{12n^2}\tilde{\delta}_{\mu\nu} + \ n.p.
\tag{3.15}
$$

which agrees with (3.12).

Let us consider the large $N$ limit where we fix $n^2/f^4 N$ as the 't Hooft couplings. They are appropriate 't Hooft couplings in generic NC gauge theories on $S^2$. It is because we obtain a factor $n/N$ at each loop due to $6j$ symbols in the interaction vertices. The other factor $n/f^4$ is generic in $U(n)$ gauge theory. In such a large $N$ limit, the correlators scale as follows to all orders

$$
\frac{1}{N}Tr A_\mu A_\nu \sim N^{\frac{3}{2}}\tilde{\delta}_{\mu\nu}.
\tag{3.16}
$$

It is because the quantum corrections are small except the one loop tadpole contributions in bosonic models as we have seen in this section.

---

[6]Through this investigation, we have realized that the coefficient 12 in the third term of (3.13) was erroneously evaluated to be 10 in [19]. We have corrected this error in [21].

We can explain this scaling behavior by repeating the same argument with[19]. In this argument, the large $N$ limit of the amplitudes are estimated by their field theory counterparts. It is because we obtain 2d gauge theory in a semiclassical limit. In our case, the correlator is found to be logarithmically divergent at the one loop level in (3.3). It corresponds to the logarithmic divergence of the two point functions in field theory. Since 2d gauge theory is super renormalizable, higher loop corrections in 1PI diagrams are finite.

As for the tadpole contributions, they can be estimated by the effective action. The effective action is quadratically divergent at the one loop level in bosonic models. It is the reason why the one loop tadpole contribution (3.4) is of the same order with the tree estimates of the correlators in the large $N$ limit. The higher order contributions are suppressed by a power of $1/N^2$ due to super renormalizability of 2d gauge theory.

With $U(1)$ gauge group in supersymmetric models, we may be able to consider another scaling limit where $1/f^4 N^2$ is adopted as the 't Hooft coupling. It is because another power of $1/N$ can be gained due to the cancellations between the planar and non-planar amplitudes. In this case, we find the following scaling behavior

$$\frac{1}{N} Tr A_\mu A_\nu \sim N \tilde{\delta}_{\mu\nu},$$ (3.17)

where it is again dominated by the tree contribution. It is because the one loop corrections are $O(log(N))$ and the higher loop corrections are $O(1)$ by the power counting arguments. We therefore conclude that the order parameter on $S^2$ is always given by a unique tensor: Kronecker's $\delta$ in 3d in which $S^2$ extends. We note that the behavior of the order parameter remains the same with the classical limit.

Another conclusion we can draw in this section is that we only need to calculate 1PI diagrams in our evaluation of the gauge invariant correlators. In this procedure we expand $A_\mu$ around the quantum solutions of the effective action. In fact we can give a generic argument to justify this procedure as follows.

In order to calculate the correlators of $Tr A_\mu A_\nu$, it is sufficient to calculate the free energy of the deformed action as

$$e^{-F} = \int dA d\psi e^{-S - \eta^{\mu\nu} Tr A_\mu A_\nu},$$

$$< Tr A_\mu A_\nu > = \frac{\delta F}{\delta \eta^{\mu\nu}} \Big|_{\eta^{\mu\nu} = 0}, \cdots.$$ (3.18)

As it is well known, the free energy and the effective action are related by the Legendre

transform:

$$
\begin{aligned}
W[J] &= \Gamma[\phi] + J \cdot \phi, \\
J &= \frac{\delta\Gamma[\phi]}{\delta\phi},
\end{aligned}
\tag{3.19}
$$

where $\phi$ denotes a generic field. We can hence calculate $F = W[0]$ by minimizing the effective action $\Gamma[\phi]$. The derivations with respect to $\eta^{\mu\nu}$ are equivalent to the insertions of $TrA_\mu A_\nu$ operator into the connected or 1PI diagrams respectively.

# 4  Correlators on $S^2 \times S^2$

In this section, we compute the vacuum expectation value of the gauge invariant operators $TrA_\mu A_\nu$ on a 4 dimensional homogeneous space: fuzzy $S^2 \times S^2$. The order parameters on such a space can be evaluated in an analogous way as on $S^2$ in the preceding section. However they exhibit a different scaling behavior in the large $N$ limit due to different dimensionality. We adopt the strategy to compute the 1PI diagrams by expanding $A_\mu$ around a quantum solution which extremizes the effective action in this section.

The first option is to consider the following deformation:

$$
S_{IIB} + \frac{i}{3}f_{\mu\nu\rho}Tr[A_\mu, A_\nu]A_\rho,
\tag{4.1}
$$

where $f_{\mu\nu\rho}/f$ are the structure constants of $SU(2) \times SU(2)$. A fuzzy $S^2 \times S^2$ is a classical solution of this model. At the tree level, the order parameter is evaluated as

$$
\frac{f^2}{N}Trp_\mu p_\nu = \frac{f^2}{3}l(l+1)\breve{\delta}_{\mu\nu} \sim \frac{f^2 N}{12n}\breve{\delta}_{\mu\nu},
\tag{4.2}
$$

where $N = n(2l+1)^2$. We here assume that the both $S^2$ are of the identical size. The symbol $\breve{\delta}_{\mu\nu} \equiv f_{\mu\rho\sigma}f_{\nu\rho\sigma}/2f^2$ is Kronecker's $\delta$ in the 6 dimensional sub-space in which $S^2 \times S^2$ extends. (4.2) is consistent with the fact that the eigenvalues of matrices are distributed as $S^2 \times S^2$ in the 6d space with the radius $fl$ at the classical level.

At the one loop level, we obtain

$$
\begin{aligned}
\frac{f^2}{N} < Tra_\mu a_\nu > &= \frac{n^2}{f^2 N}\sum_{j,p}\frac{(2j+1)(2p+1)}{j(j+1)+p(p+1)}\delta_{\mu\nu} \\
&\sim \frac{n}{f^2}2log(2)\delta_{\mu\nu}.
\end{aligned}
\tag{4.3}
$$

Since we identify $n^2/f^4 N$ as the 't Hooft coupling, it is of the same order with the tree contribution (4.2).

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In order to evaluate the two loop contribution, we first evaluate the one loop self energy of gauge field. This evaluation is reported in appendix C. The order parameter $< TrA_\mu A_\nu >$ itself is quadratically divergent by power counting on 4 dimensional spaces like $S^2 \times S^2$. In this case, we can safely ignore non-planar contributions since they are suppressed by a power of $N$. The leading part of the gauge field self-energy (C.5) is

$$\frac{1}{2} < a^\mu | \frac{1}{Q^2 R^2} \Big( 4P^2 \delta_{\mu\nu} - 4P_\mu P_\nu \Big) | a^\nu >_p .$$ 
(4.4)

The two loop contribution to the correlator follows from this estimate as

$$\frac{f^2}{N} < Tra_\mu a_\nu >$$
$$= -\frac{n^3}{f^6 N} (4\delta_{\mu\nu} - \frac{2}{3}\breve{\delta}_{\mu\nu}) < \frac{1}{P^2 Q^2 R^2} >_p,$$
(4.5)

where

$$< X >_p = \sum_{j_i, p_i, m_i, q_i} \Psi^*_{123} X \Psi_{123},$$
$$\Psi_{123} = TrY_{j_3 m_3} Y_{j_2 m_2} Y_{j_1 m_1} TrY_{p_3 q_3} Y_{p_2 q_2} Y_{p_1 q_1}.$$
(4.6)

We also need to determine $< A_\mu >= \beta p_\mu$ by minimizing the two loop effective action. We can again read off the relevant effective action from our previous results[19][21] as

$$(\frac{1}{2}\beta^4 - \frac{2}{3}\beta^3 f)\frac{N^2}{2n}$$
$$- \frac{n^3}{\beta^4} \Big( 16F_4^p + 8(\frac{f^2}{\beta^2})F_3^p - 12(1 - \frac{f}{\beta})F_3^p + 32(1 - \frac{f}{\beta})(F_3^p - F_4^p) + 24(1 - \frac{f}{\beta})F_3^p \Big),$$
(4.7)

where the first and second line represent the tree and two loop contributions respectively. As in $S^2$ case: (3.13), the second term proportional to $(f^2/\beta^2)$ denotes the contribution from the Myers vertices. The third and fourth terms come from gauge and fermionic contributions and the last term comes from the Myers and gauge cubic vertices.

We have ignored the one loop contribution $O(log(N))$ in comparison to the tree and the two loop contributions of $O(N)$ in the large $N$ limit where $n^2/f^4 N$ is fixed. It is a natural 't Hooft coupling in 4d NC gauge theory as well since we obtain a factor of $n/N$ due to the $6j$ symbols in the interaction vertices at each loop. Since 4d gauge theory is renormalizable, the over all degree of divergence remains the same to all orders.

The deviation from the classical solution can be estimated by minimizing (4.7) with respect to $\beta$ to the leading order of $n^2/f^4 N$ as

$$1 - \frac{f}{\beta} = -\frac{n^4}{f^8 N^2}(96 F_4^p + 4 F_3^p). \tag{4.8}$$

Since $F_3^p$ and $F_4^p$ are $O(1)$ in the large $N$ limit, the two loop corrections are finite and small as long as the 't Hooft coupling is small. We have further checked that the two loop tadpole contributions are consistent with this result in appendix C.

Alternatively we can evaluate the correlators on $S^2 \times S^2$ in IIB matrix model itself. In fact it has been found that a fuzzy $S^2 \times S^2$ is a consistent solution of IIB matrix model at the two loop level[19][21]. Furthermore the most symmetric case where the both $S^2$ possess the identical radii is favored by the effective action. It constitutes another evidence that 4 dimensionality of space-time may be dynamically explained in IIB matrix model [27][28][29]. In this case, $< A_\mu > = f p_\mu$ is determined by minimizing the two loop effective action. Expanding $A_\mu$ around the quantum solution, the order parameter is given by the total of (4.2), (4.3) and (4.5):

$$\frac{f^2 N}{12 n} \breve{\delta}_{\mu\nu} + \frac{n}{f^2} 2 log(2) \delta_{\mu\nu} - \frac{n^3}{f^6 N}(4 \delta_{\mu\nu} - \frac{2}{3} \breve{\delta}_{\mu\nu}) F_3^p$$
$$= \sqrt{N}(\frac{1}{6\lambda} \breve{\delta}_{\mu\nu} + \lambda log(2) \delta_{\mu\nu} - \lambda^3 \frac{1}{2}(\delta_{\mu\nu} - \frac{1}{6} \breve{\delta}_{\mu\nu}) F_3^p), \tag{4.9}$$

where $\lambda^2 = 4 n^2/f^4 N$.

However we have found $\lambda^2 = 1/\sqrt{F_P^3} \sim 0.55$ at the extremum of the two loop effective action. Therefore the two loop corrections in (4.9) are not small compared to the tree and the one loop corrections. Since a naive perturbation theory is not reliable, we adopt a resummation procedure to use the one loop exact propagator for gauge field. In this resummation, we insert the one loop gauge field self-energy into the gauge field propagator to all orders.

Let us investigate the one loop gauge field self-energy in (4.4) which is of the form $(P^2 \delta_{\mu\nu} - P_\mu P_\nu)\omega(P)$:

$$\omega(j_1, p_1) = \frac{4}{f^4} \sum_{j_2, j_3, p_2, p_3} \frac{(2 j_2 + 1)(2 p_2 + 1)(2 j_3 + 1)(2 p_3 + 1)}{(j_2(j_2 + 1) + p_2(p_2 + 1))(j_3(j_3 + 1) + p_3(p_3 + 1))}$$
$$\times \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ l & l & l \end{array} \right\}^2 \left\{ \begin{array}{ccc} p_1 & p_2 & p_3 \\ l & l & l \end{array} \right\}^2, \tag{4.10}$$

where we refer [30] for $6j$ symbols. In the large $N$ limit, we can adopt the Wigner approximation of $6j$ symbols to estimate the above amplitude[21]. In such an approximation, it

reduces to the following integral

$$\omega = \frac{1}{f^4 l^2 \pi^2} \int_0^4 dY dZ dM dN \frac{1}{(Y+M)(Z+N)}$$

$$\times \frac{1}{\sqrt{2XY + 2YZ + 2ZX - X^2 - Y^2 - Z^2 - XYZ}}$$

$$\times \frac{1}{\sqrt{2LM + 2MN + 2NL - L^2 - M^2 - N^2 - LMN}}, \tag{4.11}$$

where $X = j_1^2/l^2, L = p_1^2/l^2$. We can argue that this approximation is exact in the large $N$ limit except at vanishingly small $X$ and $L$. However such a region does not concern us since it does not contribute to the quadratically divergent order parameter.

Integrating $Z$ and $M$, we obtain

$$\omega = \lambda^2 \int_0^4 dY dN$$

$$\times \frac{1}{\sqrt{X^2 + Y^2 + N^2 + 2XN + 2YN - 2XY - XYN}}$$

$$\times \frac{1}{\sqrt{L^2 + N^2 + Y^2 + 2YL + 2YN - 2LN - YLN}}. \tag{4.12}$$

For small $X, L$, the self-energy logarithmically diverges:

$$\omega \sim \lambda^2 log(\frac{4}{X+L}) \tag{4.13}$$

as in Figure 1 where we plot $\omega$ as a function of $X = L$. A full $X, L$ dependence of $\omega$ is plotted in Figure 2.

In terms of $\omega$, the one loop exact propagator to the leading order in $1/P^2$ expansion is

$$\frac{1}{P^2} \frac{1}{1+\omega} \delta_{\mu\nu} + \frac{P_\mu P_\nu}{P^4} \frac{\omega}{1+\omega}. \tag{4.14}$$

The order parameter is given in this resummation scheme as

$$\sqrt{N} \frac{1}{6\lambda} \breve{\delta}_{\mu\nu} + \frac{f^2}{N} < Tr a_\mu a_\nu >$$

$$= \sqrt{N} \frac{1}{6\lambda} \breve{\delta}_{\mu\nu} + \frac{1}{f^2 N} \sum_{j,p} \frac{(2j+1)(2p+1)}{j(j+1)+p(p+1)} \frac{1}{1+\omega(j,p)} \delta^{\mu\nu}$$

$$+ \frac{1}{6f^2 N} \sum_{j,p} \frac{(2j+1)(2p+1)}{j(j+1)+p(p+1)} \frac{\omega(j,p)}{1+\omega(j,p)} \breve{\delta}_{\mu\nu}. \tag{4.15}$$

In our approximation, it reduces to the following integrals

$$\sqrt{N} \frac{1}{6\lambda} \breve{\delta}_{\mu\nu} + \sqrt{N} \frac{\lambda}{2} \int_0^4 dX dL \frac{1}{X+L} \frac{1}{1+\omega(X,L)} \delta_{\mu\nu}$$

$$+ \sqrt{N} \frac{\lambda}{12} \int_0^4 dX dL \frac{1}{X+L} \frac{\omega(X,L)}{1+\omega(X,L)} \breve{\delta}_{\mu\nu}. \tag{4.16}$$
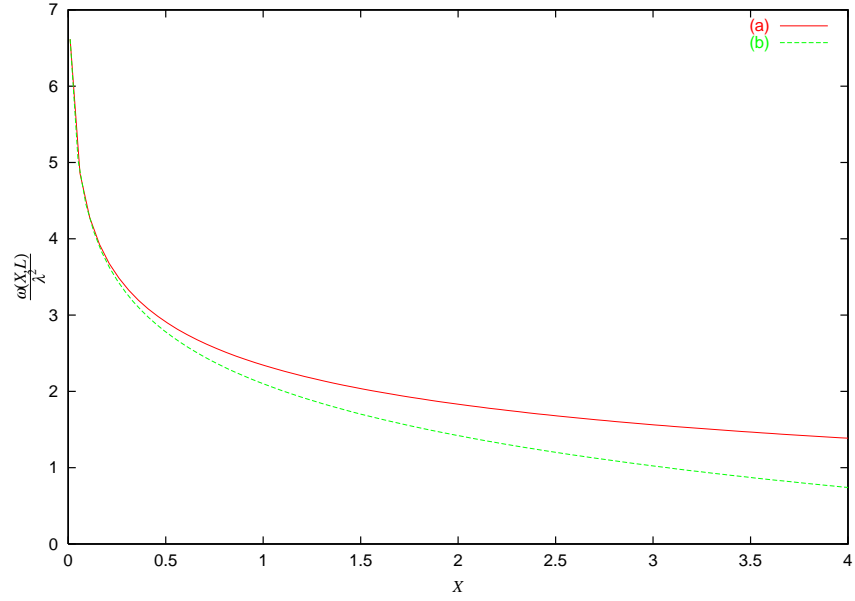
14

Figure 1: Plot of $\omega(X, L = X)$ against $X$:
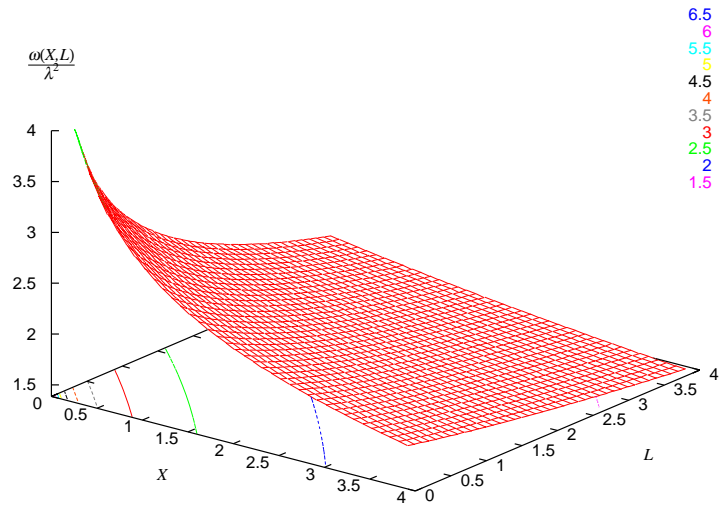Line (a) is $\frac{\omega(X, L=X)}{\lambda^2}$ and line (b) is $0.98 \ln(4/X) + 0.74$.



Figure 2: Plot of $\omega(X, L)$ against $X$ and $L$

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We can numerically estimate the above integrals with $\lambda^2 \sim 0.55$ as

$$\sqrt{N}\Big(0.48\breve{\delta}_{\mu\nu} + 0.91\delta_{\mu\nu}\Big). \tag{4.17}$$

Although this estimate is not exact, it is a universal conclusion that the gauge invariant correlator $< \frac{1}{N}TrA_\mu A_\nu >$ scales as $\sqrt{N}$ in 4 dimensional homogeneous spaces $S^2 \times S^2$ in the large $N$ limit. It is because we can estimate the scaling behavior of the correlators to all orders by the power counting arguments. In 4d, gauge theory is renormalizable. Therefore the over all degree of the divergence of the correlators remains the same to all orders. The scaling behavior of the order parameter ($\sqrt{N}$) is the reflection of the fact that it is quadratically divergent. Hence this scaling behavior is valid to all orders with our identification of the 't Hooft couplings.

The order parameter consists of the two independent tensors $\breve{\delta}_{\mu\nu}$ and $\delta_{\mu\nu}$. The first part can be interpreted as a distribution of $S^2 \times S^2$ extending in the 6 dimensional space just like our background. Since it is $O(R^2)$ in term of its radius $R$, we may identify $R \sim N^{1/4}$ from (4.17). The non-commutativity scale (volume of a quantum) is set by the 't Hooft couplings.

The second term proportional to $\delta_{\mu\nu}$ solely arises from quantum fluctuations. The identical $N$ dependence indicates that it also represents a 4 dimensional distribution. However $\delta_{\mu\nu}$ implies that it extends in 10 dimensions. We interpret it as the contributions from a fractal whose fractal dimension is 4. A concrete example is the branched polymers which have appeared in a previous study of IIB matrix model around commutative backgrounds[27]. Furthermore we have argued that 4d fuzzy homogeneous spaces are smoothly connected with the branched polymers[19]. The large $N$ scaling behavior of the gauge invariant correlator (4.17) is consistent with our viewpoint.

## 5    Connected two point functions

In this section, we investigate the connected two point functions of the gauge invariant operators $\frac{1}{N}TrA_\mu A_\nu$. In particular we are interested in the operators ($\mu \neq \nu$) with no vacuum expectation values. Let us consider the connected two point functions of them on $S^2$. It is most convenient to expand the functions (or matrices) in terms of spherical harmonics $Y_{lm}$. Since $p_\mu$ can be identified with $Y_{1\mu}$, this operator can be regarded as the gauge invariant completion of $\text{Tr}(Y_{l\mu}a_\nu + Y_{l\nu}a_\mu)$, namely a Wilson line on $S^2$ which carries the minimum angular momentum[25][26].

At the tree level, the connected two point functions can be evaluated as

$$< \frac{f^2}{N} Tr(p_\mu a_\nu + a_\mu p_\nu) \frac{f^2}{N} Tr(p_\rho a_\sigma + a_\rho p_\sigma) >$$
$$= \frac{N}{12n^2} \Big( \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} \Big). \tag{5.1}$$

At the one loop level, there occurs the following contribution

$$< \frac{f^2}{N} Tr a_\mu a_\nu \frac{f^2}{N} Tr a_\rho a_\sigma >$$
$$= \frac{n^2}{f^4 N^2} \sum_j \frac{2j+1}{(j(j+1))^2} \Big( \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} \Big)$$
$$= \frac{n^2}{f^4 N^2} \Big( \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} \Big). \tag{5.2}$$

This contribution vanishes in comparison to the tree contribution (5.1) in the large $N$ limit where either $n^2/f^4 N$ or $1/f^4 N^2$ is fixed. We do find nonvanishing contributions to this correlator at the one loop level through the quantum corrections of the gauge field propagator. In the case of the 3D bosonic matrix model, the one loop self-energy of gauge field with small angular momentum scales as in (2.14). With the self-energy insertion into the propagator, the two point function behaves as

$$- \frac{N}{12n^2} \Big( \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} \Big) \frac{2}{3} \tilde{\lambda}^2 log(N/n), \tag{5.3}$$

where $\tilde{\lambda}^2 = n^2/f^4 N$. We thus find that the two point functions $NG_2$ on $S^2$ exhibits the logarithmic scaling violation at the one loop level. Since we can trace it to the logarithmically divergent one loop mass term, these logarithms may be resummed as

$$G_2 \sim \frac{1}{2 + \frac{4}{3}\tilde{\lambda}^2 log(N/n)} \frac{1}{6n^2} \Big( \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} \Big). \tag{5.4}$$

In the case of supersymmetric models, the corresponding logarithmic scaling violations are absent since quantum corrections are finite to all orders in the large $N$ limit.

We next investigate the connected two point functions of these Wilson lines on $S^2 \times S^2$ in IIB matrix model. At the tree level,

$$< \frac{f^2}{N} Tr(p_\mu a_\nu + a_\mu p_\nu) \frac{f^2}{N} Tr(p_\rho a_\sigma + a_\rho p_\sigma) >$$
$$= \frac{1}{12n} \Big( \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} \Big). \tag{5.5}$$

At the one loop level, there occurs the following contribution

$$
\begin{aligned}
&< \frac{f^2}{N} Tr a_\mu a_\nu \frac{f^2}{N} Tr a_\rho a_\sigma > \\
&= \frac{n^2}{f^4 N^2} \sum_{j,p} \frac{(2j+1)(2p+1)}{(j(j+1)+p(p+1))^2} \Big( \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} \Big) \\
&= \frac{n^2}{f^4 N^2} log(N/n) \Big( \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} \Big).
\end{aligned}
\tag{5.6}
$$

This term is suppressed by $1/N$ in comparison to the tree term (5.5) in the large $N$ limit where $n^2/f^4 N$ is fixed. Therefore the nontrivial contributions in the large $N$ limit also come from the quantum corrections of the gauge field propagator on $S^2 \times S^2$.

With the one loop self-energy estimation (4.13), we obtain

$$
-\frac{1}{12n}\lambda^2 log(N/n)\Big( \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} \Big).
\tag{5.7}
$$

We thus find the logarithmic scaling violation of the two point functions $G_4$ on $S^2 \times S^2$ also. We can trace them to the wave function renormalization of gauge field. It is therefore likely that these logarithms may be summed by the renormalization group. Let us contemplate to renormalize the gauge field as

$$
\begin{aligned}
A_\mu &= p_\mu + Z_{j,p} a_\mu^{j,p} Y_j \otimes Y_p, \\
Z_{j,p}^2 &= 1 - \lambda^2 log\Big( (N/n)(j^2 + p^2) \Big).
\end{aligned}
\tag{5.8}
$$

It is then natural to renormalize the two point functions of the Wilson lines as

$$
G_4^R = Z^{-2} G_4.
\tag{5.9}
$$

By assuming $G_4^R$ possesses a finite large $N$ limit, we obtain the following renormalization group equation for bare $G_4$

$$
N\frac{\partial}{\partial N} G_4 = -\lambda^2 G_4.
\tag{5.10}
$$

This equation suggests that $G_4$ scales as $(N/n)^{-\lambda^2}$ in the large $N$ limit with $\lambda^2$ being fixed.

It is clear that we need to renormalize NC gauge theory on homogeneous spaces to determine the scaling behavior of the Wilson lines. Admittedly our investigation in this section is exploratory and a more detailed investigation on this problem is necessary. We also need to study the effects of the non-planar contributions.

# 6  Conclusions and Discussions

In this paper, we have investigated correlators of $Tr A_\mu A_\nu$ in matrix models of homogeneous spaces such as $S^2$ and $S^2 \times S^2$. It measures the geometry of Euclidean space-time on which NC gauge theory is realized. Since space-time and matter are unified in matrix models, the structure of space-time in general receives quantum corrections. This feature also suggests a close connection between NC gauge theory and quantum gravity.

We have investigated these correlators in matrix models up to two loop level. We have developed a procedure to evaluate them through 1PI diagrams. We have determined the large $N$ scaling behavior of the geometric order parameters to all orders.

On 2 dimensional spaces $S^2$, we have found that the expectation values of them are given by a unique tensor just like in the classical limit. This fact shows that the eigenvalues of matrices are always distributed as $S^2$. On 4 dimensional spaces $S^2 \times S^2$, we have found that these order parameters consist of two independent tensors. Although the first tensor is due to a distribution of $S^2 \times S^2$ , the second tensor indicates a distributions of a 4d fractal. We conclude that fuzzy $S^2 \times S^2$ acquires a 4 dimensional fractal structure due to NC gauge theory on it in contrast to fuzzy $S^2$.

The operators with vanishing vacuum expectation values are the Wilson lines which carry the minimum momentum. We have investigated the two point functions of these Wilson lines on $S^2$ and $S^2 \times S^2$. We have found logarithmic scaling violations (or finite anomalous dimensions) of the Wilson lines. We note that it is a suppression effect for low multipoles which might be relevant to such a tendency observed in the cosmic microwave background. We believe that our procedure developed in this paper should be useful to study more generic Wilson lines. We hope that such investigations will further elucidate gravitational aspects of NC gauge theory.

In the context of IIB matrix model, fuzzy $S^2 \times S^2$ solutions may be interpreted as metastable D3-branes. We are most interested in the ultimate ground state into which these branes eventually decay. This is an analogue of tachyon condensation problem of unstable D-branes in IIB matrix model[31]. We also hope to make progress toward understanding this fundamental question.

## Acknowledgments

# Appendix A

In this appendix, we evaluate the one loop gauge field self-energy in $D$ dimensional bosonic reduced models with Myers term. We list the contributions from different diagrams separately. We only show planar contributions since non-planar contributions can be obtained from them as in (2.6)

- From cubic gauge couplings:

$$\frac{1}{2} < a^\mu | \frac{1}{Q^2 R^2} \Big( 4 P_\nu P_\mu - (D-2)(Q_\nu Q_\mu + R_\nu R_\mu)$$
$$+ (D-1)(Q_\mu R_\nu + Q_\nu R_\mu) - (4P^2 + Q^2 + R^2)\delta_{\mu\nu} \Big) | a^\nu >_p .$$

(A.1)

- From quartic gauge coupling:

$$\frac{D-1}{2} < a^\mu | \frac{1}{Q^2 R^2} \Big( (Q^2 + R^2)\delta_{\mu\nu} \Big) | a^\nu >_p .$$

(A.2)

- From ghost:

$$-\frac{1}{2} < a^\mu | \frac{1}{Q^2 R^2} \Big( Q_\mu R_\nu + Q_\nu R_\mu \Big) | a^\nu >_p .$$

(A.3)

- From Myers terms:

$$-\frac{8}{2} < a^\mu | \frac{1}{Q^2 R^2} \tilde{\delta}_{\mu\nu} | a^\nu >_p .$$

(A.4)

where $\tilde{\delta}_{\mu\nu} \equiv \epsilon_{\mu\rho\sigma} \epsilon_{\nu\rho\sigma}/2$ is the projector into the 3 dimensional sub-space in which $S^2$ extends.

- From Myers term and cubic gauge coupling:

$$\frac{12}{2} < a^\mu | \frac{1}{Q^2 R^2} i\epsilon_{\mu\nu\rho} P^\rho | a^\nu >_p .$$

(A.5)

In total we obtain (2.13) in section 2.

We further list the planar part of the ghost and fermion self-energy at the one loop level for completeness.

- The ghost self energy:

$$- < b | \frac{1}{Q^2 R^2} P^2 | c >_p .$$

(A.6)

- The fermion self energy:

$$-4 < \bar{\psi}| \frac{1}{Q^2 R^2} P \cdot \Gamma |\psi >_p$$

$$-i < \bar{\psi}| \frac{1}{(Q^2)^2 R^2} \epsilon_{\mu\nu\rho} \Gamma^{\mu\nu\sigma} Q^\rho Q^\sigma |\psi >_p -i < \bar{\psi}| \frac{1}{Q^2 (R^2)^2} \epsilon_{\mu\nu\rho} \Gamma^{\mu\nu\sigma} R^\rho R^\sigma |\psi >_p .$$

$$(A.7)$$

# Appendix B

In this appendix, we evaluate the tadpole contributions to the following correlator on $S^2$:

$$\frac{f^2}{N} < Tr p_\mu a_\nu + a_\mu p_\nu > . \tag{B.1}$$

The relevant cubic vertices in a deformed IIB matrix model are

$$f^4 Tr \Big( P_\mu a_\nu [a_\mu, a_\nu] - \frac{i}{3} \epsilon_{\mu\nu\rho} a_\mu [a_\nu, a_\rho] + \frac{1}{2} \bar{\psi} \Gamma_\mu [a^\mu, \psi] + P_\mu b [a_\mu, c] \Big). \tag{B.2}$$

By attaching the one loop self-energy of gauge field (2.16) to the cubic vertices, we evaluate the pure gauge field contribution to (B.1) as

$$\frac{n^3}{f^6 N} \tilde{\delta}_{\mu\nu} \Big( < \frac{12}{Q^2 R^2} >_p -\frac{4}{3} < \frac{1}{P^2 Q^2 R^2} >_p -\frac{104}{3} < \frac{1}{P^2 Q^2 R^2} >_p \Big) + n.p.. \tag{B.3}$$

We note that the contributions from the longitudinal part of the gauge field self-energy (the second terms in (3.10) and (B.3)) to the gauge invariant correlator cancel out each other.

The remaining contributions are the followings.

- The ghost contribution
$$\frac{n^3}{f^6 N} \tilde{\delta}_{\mu\nu} \frac{1}{3} < \frac{1}{Q^2 R^2} >_p + n.p.. \tag{B.4}$$

- The fermion contribution
$$\frac{n^3}{f^6 N} \tilde{\delta}_{\mu\nu} \Big( -\frac{64}{3} < \frac{1}{Q^2 R^2} >_p -\frac{64}{3} < \frac{1}{P^2 Q^2 R^2} >_p \Big) + n.p.. \tag{B.5}$$

- The quartic gauge coupling contribution
$$\frac{n^3}{f^6 N} \tilde{\delta}_{\mu\nu} < \frac{9}{Q^2 R^2} >_p + n.p.. \tag{B.6}$$

In total we obtain (3.12) in section 3.

# Appendix C

In this appendix, we evaluate the planar part of the one loop gauge field self-energy on $S^2 \times S^2$. We subsequently carry out a perturbative evaluation of the tadpole contributions to the following order parameter on $S^2 \times S^2$:

$$\frac{f^2}{N} < Tr p_\mu a_\nu + a_\mu p_\nu > \qquad . \tag{C.1}$$

The contribution to the gauge field self-energy from the gauge sector is

$$\frac{1}{2} < a^\mu | \frac{1}{Q^2 R^2} \Big( 4 P_\mu P_\nu - 4 P^2 \delta_{\mu\nu} - 8 \breve{\delta}_{\mu\nu} + 8 i f_{\mu\nu\rho} P^\rho$$
$$+ (D-2)(Q_\mu R_\nu + Q_\nu R_\mu - Q_\nu Q_\mu - R_\nu R_\mu + (Q^2 + R^2) \delta_{\mu\nu}) \Big) | a^\nu >_p, \tag{C.2}$$

where in this context

$$< a^\mu | X | a^\nu >_p \; = \; \sum_{j_i, p_i, m_i, q_i} \Psi^*_{a^\mu 23} X \Psi_{a^\nu 23},$$

$$\Psi_{a^\mu 23} \; = \; \sum_{j_1 m_1 p_1 q_1} a^\mu_{j_1 m_1 p_1 q_1} Tr Y_{j_3 m_3} Y_{j_2 m_2} Y_{j_1 m_1} Tr Y_{p_3 q_3} Y_{p_2 q_2} Y_{p_1 q_1}. \tag{C.3}$$

The fermion sector contribution has the form as the following

$$\frac{16}{2} < a_\mu | \frac{n}{P^2 R^2} \Big[ Q_\mu R_{\nu+} Q_\nu R_\mu - Q \cdot R \, \delta_{\mu\nu}$$

$$-i \bar{f}_{\mu\nu\rho} \left( \frac{\bar{Q}_\rho (\tilde{Q} \cdot \tilde{R}) + \bar{R}_\rho \bar{Q}^2}{Q^2} + \frac{\bar{Q}_\rho \bar{R}^2 + \bar{R}_\rho (\tilde{Q} \cdot \tilde{R})}{R^2} \right)$$

$$-i \tilde{f}_{\mu\nu\rho} \left( \frac{\tilde{Q}_\rho (\bar{Q} \cdot \bar{R}) + \tilde{R}_\rho \bar{Q}^2}{Q^2} + \frac{\tilde{Q}_\rho \bar{R}^2 + \tilde{R}_\rho (\bar{Q} \cdot \bar{R})}{R^2} \right)$$

$$- \frac{1}{Q^2 R^2} \Big( (\bar{Q}^2 \bar{R}^2 - \tilde{Q}^2 \tilde{R}^2) \bar{\delta}_{\mu\nu} + (\tilde{Q}^2 \tilde{R}^2 - \bar{Q}^2 \bar{R}^2) \tilde{\delta}_{\mu\nu}$$

$$\bar{Q}_\mu \tilde{Q}_\nu + \bar{R}_\mu \tilde{R}_\nu - \delta_{ij} (\bar{Q}^2 \bar{R}^2 + \tilde{Q}^2 \tilde{R}^2 + 2 (\bar{Q} \cdot \bar{R})(\tilde{Q} \cdot \tilde{R})) \Big) \Big] | a_\nu >_p, \tag{C.4}$$

where $\bar{Q}_\mu, \bar{R}_\mu$ and $\tilde{Q}_\mu, \tilde{R}_\mu$ denote the components of the operators in the first and second three dimensional subspaces. $\delta_{ij}$ is the Kronecker's $\delta$ in the remaining dimensions.

We restrict our attention in the case of D=10 for simplicity. Using (C.4) we obtain the 1-loop gauge field self-energy on $S^2 \times S^2$:

$$\frac{1}{2} < a_\mu | \frac{n}{Q^2 R^2} \Big[ 4 P_\mu P_\nu - 4 P^2 \delta_{\mu\nu} - 8 \breve{\delta}_{\mu\nu} + 8 i f_{\mu\nu\rho} P^\rho$$

$$-16i\bar{f}_{\mu\nu\rho}\left(\frac{\bar{Q}_\rho(\tilde{Q}\cdot\tilde{R})+\bar{R}_\rho\bar{Q}^2}{Q^2}+\frac{\bar{Q}_\rho\bar{R}^2+\bar{R}_\rho(\tilde{Q}\cdot\tilde{R})}{R^2}\right)$$

$$-16i\tilde{f}_{\mu\nu\rho}\left(\frac{\tilde{Q}_\rho(\bar{Q}\cdot\bar{R})+\tilde{R}_\rho\bar{Q}^2}{Q^2}+\frac{\tilde{Q}_\rho\bar{R}^2+\tilde{R}_\rho(\bar{Q}\cdot\bar{R})}{R^2}\right)$$

$$-\frac{16}{Q^2R^2}\Big((\bar{Q}^2\bar{R}^2-\tilde{Q}^2\tilde{R}^2)\bar{\delta}_{\mu\nu}+(\tilde{Q}^2\tilde{R}^2-\bar{Q}^2\bar{R}^2)\tilde{\delta}_{\mu\nu}$$

$$\bar{Q}_\mu\tilde{Q}_\nu+\bar{R}_\mu\tilde{R}_\nu-\delta_{ij}(\bar{Q}^2\bar{R}^2+\tilde{Q}^2\tilde{R}^2+2(\bar{Q}\cdot\bar{R})(\tilde{Q}\cdot\tilde{R}))\Big)\bigg]|a_\nu>_p . \tag{C.5}$$

Now we can evaluate the order parameter (C.1) at the 2-loop level. There are contributions from cubic gauge, ghost, fermion, quartic gauge couplings which correspond to (B.3)-(B.6). In total we obtain

$$\frac{n^3}{f^6N}\breve{\delta}_{\mu\nu}\left(-12F_3^p-\frac{32}{3}F_4^p+\frac{64}{3}G^p\right) , \tag{C.6}$$

where

$$F_4^p = 2\left\langle\frac{\bar{Q}^2\bar{R}^2+\tilde{Q}^2\tilde{R}^2+2(\bar{Q}\cdot\bar{R})(\tilde{Q}\cdot\tilde{R})}{P^2Q^4R^4}\right\rangle_p ,$$

$$G^p = \left\langle\frac{1}{P^2Q^4R^2}\Big[(\bar{P}\cdot\bar{R})\bar{Q}^2+(\bar{P}\cdot\bar{Q})(\tilde{Q}\cdot\tilde{R})+(\tilde{P}\cdot\tilde{R})\bar{Q}^2+(\tilde{P}\cdot\tilde{Q})(\bar{Q}\cdot\bar{R})\Big]\right\rangle_p$$

$$= \frac{1}{2}F_3^p-\frac{1}{2}F_4^p . \tag{C.7}$$

Finally we obtain

$$\frac{f^2}{N}<Trp_\mu a_\nu+a_\mu p_\nu> = -\frac{n^3}{f^6N}\frac{\breve{\delta}_{\mu\nu}}{3}(4F_3^p+96F_4^p) . \tag{C.8}$$

The result (C.8) is indeed consistent with (4.8).

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